Stability of Vertex Fixed Points and Applications

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Abstract

In the Infinite Population Simple Genetic Algorithm, stability of fixed points is considered when mutation is zero. The analysis is based on the spectrum of the differential of the mapping which defines the transition from one generation to the next. Based on a simple formula for this spectrum, fully nondeceptive functions having exponentially many non-optimal fixed points are constructed.

1 INTRODUCTION

This paper is concerned with properties of a function $\mathcal{G}$ (defined in the next section) which can be regarded as answering the following fundamental questions for a simple genetic algorithm:

1. What is the *exact sampling distribution* describing the formation of the next generation?
2. What is the *expected next generation*?
3. In the limit, as population size grows, what is the *transition function* which maps from one generation to the next?

For each of these questions, the answer provided by $\mathcal{G}$ is exact. In a sense, $\mathcal{G}$ *is a GA*: anything that ever could be proved about the simple genetic algorithm (for arbitrary population sizes, finite or infinite) corresponds to some property of $\mathcal{G}$. It is not unnatural to refer to $\mathcal{G}$ as the "Infinite Population Simple Genetic Algorithm" since, by answering the

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third question above, it is the transition function in the infinite population case. In terms of a finite population GA, an alternate interpretation is that the sequence

$$x, \ \mathcal{G}(x), \ \mathcal{G}^2(x), \ \mathcal{G}^3(x), \ \ldots$$

is essentially the most probable transient behavior from initial population x when the population size is large. An introduction to various interpretations of $\mathcal{G}$ can be found in [2].

The relationship between the finite and infinite population GA (i.e., the connection between the behavior of a finite population GA and what the corresponding relevant properties of $\mathcal{G}$ are) is an active area of research. The most complete results to date can be found in [7].

The iterative procedure $x, \ \mathcal{G}(x), \ \ldots$ is an example of a discrete dynamical system. A basic goal in the theory of dynamical systems is to understand the nature of the sequence of iterates. *Fixed points*, solutions to $\mathcal{G}(x) = x$, frequently indicate destinations towards which trajectories may converge. It is not known whether iterates of $\mathcal{G}$ converge for every initial population x. The conjecture that this is the case is the *fundamental conjecture*.

Assuming the fundamental conjecture, fixed points represent populations towards which an infinite population GA may evolve. Approximately the same may be said concerning a large finite population GA, except being constrained to occupy points in population space corresponding to its finite population size and being subject to stochastic effects (from selection, crossover, and mutation) would keep it from converging. Fixed points nevertheless locate regions within population space where a finite population GA spends much of its time. Details concerning this type of connection between finite population behavior and fixed points of $\mathcal{G}$ can be found in [5].

This paper is primarily concerned with the stability of fixed points. Roughly speaking, a stable fixed point attracts neighboring populations, while an unstable fixed point tends to repel them. An analogy is a pencil balanced on its tip. When truly balanced, it may be stable in the sense of not moving, but the slightest perturbation is expected to send it diverging towards quite a different state. A stable equilibrium is like a pendulum hanging downwards and at rest. Small perturbations will not send it off on a divergent course. While it is possible for a dynamical system to follow a trajectory leading to an unstable fixed point, that is atypical behavior. The analysis in [7] indicates that with positive mutation, there is a strong sense in which unstable fixed points may be ignored.

Although mutation has a more profound influence on GA behavior than is generally recognized [8], it also complicates analysis. For this reason, the results we consider are for the zero mutation case. Because of continuous dependence on parameters, our results may still apply when the mutation rate is low. We also assume that strings have distinct fitness, though the differences may be arbitrarily small.

An application of our stability analysis is in the last section, where "fully nondeceptive" fitness functions are constructed for which $\mathcal{G}$ has exponentially many stable fixed points. The point is not the well known fact that functions may be difficult for a GA even when they are fully nondeceptive (see sections 5 through 7 of [4] for a discussion of various GA failure modes). The purpose is to illustrate our theoretical results with a concrete application. The particular example is important by establishing just how bad things can get; it is extremal in the sense of having the maximum possible number of stable suboptimal attractors.

2 BASICS

We consider a generalization of the infinite population model of the Simple Genetic Algorithm introduced in [6]. The domain Ω is the set of length ℓ binary strings. Let $n = 2^\ell$ and note that elements of Ω correspond to integers in the range $[0, n)$. They are thereby thought of interchangeably as integers or as bit strings which are regarded as column vectors. Because of frequent use, it is convenient to let $\mathbf{1}$ denote the vector $n-1$ (which is the vector of all ones).

Let $\oplus$ denote the bitwise exclusive-or operation, and let $\otimes$ denote the bitwise and operation on Ω. For $x \in \Omega$, the ones-complement of x is denoted by $\overline{x}$. Note that $\overline{x} = \mathbf{1} \oplus x$.

If *expr* is an experssion that is either true or false, then

$$[expr] = \begin{cases} 1 & \text{if } expr \text{ is true} \\ 0 & \text{otherwise} \end{cases}$$

Let $\delta_{ij} = [i = j]$. The $n \times n$ permutation matrix whose i, j th entry is $\delta_{i \oplus k, j}$ is denoted by σ_k. Note that $(\sigma_k x)_i = x_{i \oplus k}$. The j th column of the $n \times n$ identity matrix is the vector e_j. Indexing of vectors and matrices begins with 0.

A *population* is a real-valued vector x indexed over Ω, where $\sum x_i = 1$ and $x_i \geq 0$. The probability (or proportion) of string i in population x is x_i. The set of all populations is the unit simplex Λ in R^n. The vertices of Λ correspond to populations consisting entirely of one string type.

A $n \times n$ *mixing matrix* M implements mutation and crossover. M is defined so that $x^T M x$ is the probability that the result of doing crossover and mutation based on population x is 0. Thus $M_{i,j}$ is the probability that 0 is the result produced by parents i and j.

Since this paper only considers zero mutation, we define M for that special case (the general formula can be found in [9]). Considering $k \in \Omega$ as a *crossover mask* used with parents $i, j \in \Omega$, the children are $(i \otimes k) \oplus (j \otimes \overline{k})$ and $(j \otimes k) \oplus (i \otimes \overline{k})$. We assume one child is kept (with equal probability). If χ_k denotes the probability that mask k is used, then M is given by

$$M_{i,j} = \sum_{k \in \Omega} \frac{\chi_k + \chi_{\overline{k}}}{2} [(i \otimes k) \oplus (j \otimes \overline{k}) = 0]$$

The *twist* of an $n \times n$ matrix A, denoted by A_*, has entries $(A_*)_{i,j} = A_{i \oplus j, i}$.

Proposition 2.1 *The matrix M_* is upper triangular.*

> **Proof:** If $i \otimes j \neq 0$, then either $i \otimes j \otimes k \neq 0$ which implies $i \otimes k \neq 0$, or $i \otimes j \otimes \overline{k} \neq 0$ which implies $j \otimes \overline{k} \neq 0$. In either case $(i \otimes k) \oplus (j \otimes \overline{k}) \neq 0$ and hence $M_{i,j} = 0$.
>
> Thus, $(M_*)_{i,j} = M_{i \oplus j, i}$ can be nonzero only when $(i \oplus j) \otimes i = 0$, which is equivalent to $\overline{j} \otimes i = 0$, which implies $j \geq i$. ∎

The *recombination function* $\mathcal{M} : \Lambda \longrightarrow \Lambda$ is defined by the component equations

$$e_i^T \mathcal{M}(x) = (\sigma_i x)^T M \sigma_i x = \sum_{u,v} x_u x_v M_{u \oplus i, v \oplus i}$$

Proposition 2.2 *The differential of $\mathcal{M}$ at $x \in \Lambda$ is given by $d\mathcal{M}_x = 2\sum_u \sigma_u M_* \sigma_u x_u$.*

Proof: First note that $(\sigma_u M_* \sigma_u)_{i,j} = M_{i \oplus j, u \oplus i}$. Next, the i, j th entry of $d\mathcal{M}_x$ is

$$\frac{\partial}{\partial x_j} \sum_{u,v} x_u x_v M_{u \oplus i, v \oplus i} = \sum_{u,v} (\delta_{u,j} x_v + \delta_{v,j} x_u) M_{u \oplus i, v \oplus i} = 2 \sum_u x_u M_{i \oplus j, u \oplus i}$$

∎

Assuming a *fitness function* $f : \Omega \longrightarrow R^+$, proportional selection is the mapping from Λ into Λ defined by $x \mapsto Fx/\mathbf{1}^T Fx$, where F is the $n \times n$ diagonal matrix $F_{i,j} = \delta_{i,j} f(i)$.

The *Infinite Population Simple Genetic Algorithm* is defined as the mapping $\mathcal{G} : \Lambda \longrightarrow \Lambda$ where

$$\mathcal{G}(x) = \mathcal{M}(Fx/\mathbf{1}^T Fx)$$

As indicated in the introduction, an infinite population GA can be defined in a very natural way via the limit of a finite population GA as population size increases. It follows that such a GA is deterministic (the stochastic variations average out as the population size grows) and the next generation is given by the expected next generation. This expected next generation, given current population x, is given by $\mathcal{G}(x)$ as defined above.

Proposition 2.3 *The differential of $\mathcal{G}$ at $x \in \Lambda$ is given by*

$$d\mathcal{G}_x = \frac{1}{1^T Fx} d\mathcal{M}_{\frac{Fx}{\mathbf{1}^T Fx}} FP \qquad \textit{where} \qquad P = I - x\frac{\mathbf{1}^T F}{\mathbf{1}^T Fx}$$

Proof: The differential of $h(x) = x/\mathbf{1}^T x$ is

$$dh_x = \frac{I}{\mathbf{1}^T x} - x\frac{\mathbf{1}^T}{(\mathbf{1}^T x)^2}$$

Applying the chain rule to $\mathcal{G} = \mathcal{M} \circ h \circ F$ yields

$$d\mathcal{G}_x = d\mathcal{M}_{h \circ Fx}\, dh_{Fx}\, F$$

Since $dh_{Fx} F = FP/\mathbf{1}^T Fx$, the formula for the differential follows. ∎

3 THE SPECTRUM OF $d\mathcal{G}$

A fixed point $x \in \Lambda$ of $\mathcal{G}$ is *stable* if for any neighborhood U of x, there is a neighborhood V of x such that for each $q \in V$ the trajectory $q, \mathcal{G}(q), \mathcal{G}^2(q), \ldots$ lies in U. The fixed point x is *asymptotically stable* if it is stable and all trajectories beginning in some neighborhood of x converge to x.

The *spectral radius* of a square matrix A, denoted by $\rho(A)$, is the largest modulus of the eigenvalues of A. A standard result of dynamical systems theory is that (for any differentiable map $\mathcal{G}$) if x is a fixed point of $\mathcal{G}$ and $\rho(d\mathcal{G}_x) < 1$, then x is asymptotically stable, where $\mathcal{G}$ considered as a map from R^n into itself. Moreover, x is unstable if the spectral radius is greater than 1 (see [1], for example). Therefore information about the *spectrum* of $d\mathcal{G}_x$ (its set of eigenvalues) is important to the stability of a fixed point x.

Lemma 3.1 *The matrix $\sigma_k F \sigma_k - e_0 f^T \sigma_k$ is diagonal except for row 0.*

Proof: The matrix $\sigma_k F \sigma_k$ is diagonal, and $e_0 f^T \sigma_k$ is nonzero only in row 0. ∎

Lemma 3.2 *The matrix $\sigma_k d\mathcal{G}_{e_k} \sigma_k$ is given by $\frac{2}{f_k} M_*(\sigma_k F \sigma_k - e_0 f^T \sigma_k)$.*

Proof: We will apply Proposition 2.3 at $x = e_k$. First note that $1^T F = f^T$, $Fx = f_k e_k$, and $1^T F x = f_k$. Thus

$$\sigma_k F P \sigma_k \;=\; \sigma_k F \sigma_k - \sigma_k e_k f^T \sigma_k \;=\; \sigma_k F \sigma_k - e_0 f^T \sigma_k$$

By Proposition 2.2,

$$d\mathcal{M}_{e_k} = 2 \sum (e_k)_i \sigma_i M_* \sigma_i = 2 \sigma_k M_* \sigma_k$$

so that $\sigma_k d\mathcal{M}_{e_k} \sigma_k = 2M_*$. Appealing to proposition 2.3 gives

$$\sigma_k d\mathcal{G}_{e_k} \sigma_k = \frac{1}{f_k} (\sigma_k d\mathcal{M}_{e_k} \sigma_k)(\sigma_k F P \sigma_k) = \frac{2}{f_k} M_* (\sigma_k F \sigma_k - e_0 f^T \sigma_k)$$

∎

Corollary 3.3 *If mutation is zero, the matrix $D = \sigma_k d\mathcal{G}_{e_k} \sigma_k$ has the following properties:*

1. *Column 0 of D is zero.*
2. *D is upper triangular, with diagonal $D_{i,i} = \frac{f_{k \oplus i}}{f_k} \sum_u (\mathcal{X}_u + \mathcal{X}_{\overline{u}})[u \otimes i = 0]$ for $i > 0$.*
3. *Row 0 of D is nonpositive (i.e., $D_{0,j} \leq 0$ for $1 \leq j < n$).*
4. *The other rows of D are nonnegative (i.e., $D_{i,j} \geq 0$ for $1 \leq i < n$, $i \leq j < n$).*

Proof: By Proposition 2.1, M_* is upper triangular. By Lemma 3.1, so too is $\sigma_k F \sigma_k - e_0 f^T \sigma_k$. The product of upper triangular matrices is upper triangular. Hence, by lemma 3.2, D is upper triangular.

The 0,0 th entry of both $\sigma_k F \sigma_k$ and $e_0 f^T \sigma_k$ is f_k, hence their difference is zero. Thus, column 0 of $\sigma_k F \sigma_k - e_0 f^T \sigma_k$ is zero which, by lemma 3.2, implies column 0 of D is zero.

By lemma 3.2, $D_{0,j} = 2((M_*)_{0,j} - (M_*)_{0,0}) f_{j \oplus k}/f_k = 2(M_{j,0} - M_{0,0}) f_{j \oplus k}/f_k$ for $j > 0$. But $M_{0,0} = 1$ and $M_{j,0} \leq 1$, so this quantity is nonpositive.

By lemma 3.2, $D_{i,j} = 2(M_*)_{i,j} f_{k \oplus j}/f_k = 2M_{i \oplus j, i} f_{k \oplus j}/f_k$ for $j \geq i > 0$, and this quantity is nonnegative. The diagonal entries are $2M_{0,i} f_{k \oplus i}/f_k$ for $i > 0$. ∎

Theorem 3.4 *If mutation is zero, then the spectrum of $d\mathcal{G}_{e_k}$ is given by:*

$$spec(d\mathcal{G}_{e_k}) = \left\{ \frac{f_{i \oplus k}}{f_k} \sum_u (\mathcal{X}_u + \mathcal{X}_{\overline{u}})[u \otimes i = 0] \;:\; i = 1, 2, \ldots, n-1 \right\} \bigcup \{0\}$$

Proof: Since the spectrum is invariant under conjugation, the spectrum of $d\mathcal{G}_{e_k}$ is the same as that of $\sigma_k d\mathcal{G}_{e_k} \sigma_k$. The spectrum of a triangular matrix is the set of diagonal entries. These entries are given by Corollary 3.3. ∎

4 STABILITY IN THE SIMPLEX

Theorem 3.4 can be used to compute $\rho(d\mathcal{G}_{e_k})$ and thus characterize the stability at e_k of $\mathcal{G}$ as a map from R^n to R^n. One might wonder however if a fixed point of $\mathcal{G}$ could be unstable in this sense, but stable when the domain of $\mathcal{G}$ is restricted to Λ. Among other things, this section shows that cannot happen.

Before proceeding, we review some facts concerning the Jordan canonical form. A $m \times m$ *simple Jordan submatrix* is identical to a constant times the identity matrix, except that the subdiagonal consists of 1's. For example,

$$\begin{pmatrix} a & 0 & 0 \\ 1 & a & 0 \\ 0 & 1 & a \end{pmatrix}$$

is a 3×3 simple Jordan submatrix. Given any square matrix A, there exists a similarity transformation P such that $J = P^{-1}AP$ is in *Jordan canonical form*: it is block diagonal with simple Jordan submatrices along the diagonal. The diagonal entries of a block are eigenvalues and the columns of P corresponding to the columns occupied by a block (in J) form a basis for a space invariant under A.

We apply the Jordan canonical form to $d\mathcal{G}_{e_k}$, dividing the simple Jordan blocks into two categories: *stable* blocks corresponding to eigenvalues λ with $|\lambda| \leq 1$, and *unstable* blocks corresponding to eigenvalues λ with $|\lambda| > 1$. The columns of P (the matrix of the similarity transformation) corresponding to stable blocks form a basis for the *stable space* S. The columns corresponding to unstable blocks form a basis for the *unstable space* U. Thus R^n is decomposed into the direct sum of S and U, and each is invariant under $d\mathcal{G}_{e_k}$. Let $\pi_S : R^n \longrightarrow S$ and $\pi_U : R^n \longrightarrow U$ be the projections into these subspaces.

Let $\|\cdot\|_S$ and $\|\cdot\|_U$ denote norms on S and U respectively. Given $\theta > 0$, we define the *s-region* $r(\theta) = \{x \in R^n : \|\pi_S(x)\|_S < \theta\|\pi_U(x)\|_U\}$. Note that $r(\theta)$ depends on the choice of norms on U and S as well as on the parameter θ.

Theorem 4.1 *Let x be a fixed point of $f : R^n \longrightarrow R^n$ such that $\rho(df_x) > 1$. Let U and S be the unstable and stable spaces corresponding to the differential df_x. For all $\theta > 0$, there exist norms on S and U and a corresponding* s-region *$r(\theta)$ such that if V is a sufficiently small neighborhood of x, then $y \in V \cap (x + r(\theta))$ implies that $f^i(y) \notin V$ for some i.*

Proof: We first choose appropriate norms. Let $D_S = df_x|S$ and let $D_U = df_x|U$. Then $\rho(D_S) \leq 1$ and all eigenvalues of D_U are greater than 1 (thus D_U is invertible and $\rho(D_U^{-1}) < 1$). Choose norms $\|\cdot\|_S$ and $\|\cdot\|_U$ so that $\|D_U^{-1}\|_U = \beta^{-1} < 1$ and $\|D_S\|_S = \alpha < \beta$ (see for example [1]). It follows that $\|df_x s\|_S = \|D_S s\| \leq \alpha\|s\|_S$ and $\|df_x u\| = \|D_U u\|_U \geq \beta\|u\|_U$ for all $s \in S$, $u \in U$. Define the norm $\|\cdot\|$ on R^n by $\|x\| = \|s + u\| = \|s\|_S + \|u\|_U$. These norms will be assumed throughout the rest of the proof, and their subscripts will be dropped to streamline notation.

The next step is to show that for any $\theta > 0$, there exists a neighborhood V of x such that if $y \in V \cap (x + r(\theta))$ then $f(y) \in x + r(\theta)$.

Suppose $y \in V \cap (x + r(\theta))$ and let $y - x = s + u$ where $s \in S$ and $u \in U$. By the definition of the differential, $f(y) - x = df_x(y - x) + o(y - x) = df_x s + df_x u + o(y - x)$.

Choose $\eta > 0$ such that $\alpha\theta+(\theta+1)\eta \leq \theta(\beta-(\theta+1)\eta)$. Now choose the neighborhood V sufficiently small that $y \in V \Longrightarrow \|o(y-x)\| < \|y-x\|\eta$. It follows that

$$\begin{aligned}
\|\pi_S(f(y)-x)\| &\leq \|df_x s\| + \|\pi_S\,(o(y-x))\| \\
&\leq \alpha\|s\| + (\|s\| + \|u\|)\eta \\
&\leq \alpha\theta\|u\| + (\theta\|u\| + \|u\|)\eta \\
&\leq \|u\|(\alpha\theta + (\theta+1)\eta) \\
&\leq \theta\|u\|(\beta - (\theta+1)\eta) \qquad \text{by the choice of } \eta \\
&\leq \theta(\|u\|\beta - (\|s\| + \|u\|)\eta) \\
&\leq \theta(\|df_x u\| - \|y-x\|\eta) \\
&\leq \theta(\|df_x u\| - \|\pi_U(o(y-x))\|) \\
&\leq \theta\|\pi_U(f(y)-x)\|
\end{aligned}$$

Finally, we show that if $y \in V \cap (x + r(\theta))$ then $f^t(y) \notin V$ for some t. Otherwise, the trajectory $y, f(y), f^2(y), \ldots$ lies in $V \cap (x + r(\theta))$ by what has already been shown. Applying $\|\pi_U(f(y)-x)\| \geq \|u\|(\beta-(\theta+1)\eta)$ (which follows from the inequalities above) we conclude that $\|\pi_U(f^k(y)-x)\| \geq \|u\|(\beta-(\theta+1)\eta)^k$ for all k. Since η may be chosen so that $\beta-(\theta+1)\eta > 1$, this contradicts that V is a bounded neighborhood. ∎

Lemma 4.2 *For $i \neq k$, if $f_i \sum_u (\chi_u + \chi_{\overline{u}})[u \otimes (i \oplus k) = 0] > f_k$ then the stable space of $d\mathcal{G}_{e_k}$ does not intersect $\{p \in \Lambda : p_i > 0\} - e_k$.*

Proof: Abbreviate $\sigma_k d\mathcal{G}_{e_k} \sigma_k$ by D and let $p \in \Lambda$ be such that $p_i > 0$. We first show $e_i^T d\mathcal{G}_{e_k}(p-e_k) > e_i^T(p-e_k)$. By assumption, $D_{i\oplus k, i\oplus k} > 1$ (see corollary 3.3).

$$\begin{aligned}
e_i^T d\mathcal{G}_{e_k}(p-e_k) &= e_i^T \sigma_k(\sigma_k d\mathcal{G}_{e_k}\sigma_k)\sigma_k(p-e_k) \\
&= (\sigma_k e_i)^T D(\sigma_k p - e_0) \\
&= e_{i\oplus k}^T D\sigma_k p - e_{i\oplus k}^T D e_0
\end{aligned}$$

By Corollary 3.3, column 0 of D is zero. Hence $e_{i\oplus k}^T D e_0 = 0$. The term corresponding to $u = i \oplus k$ in

$$e_{i\oplus k}^T D\sigma_k p = \sum_u D_{i\oplus k, u}\, p_{u\oplus k}$$

is $D_{i\oplus k, i\oplus k}\, p_i$, which is greater than p_i (since $D_{i\oplus k, i\oplus k} > 1$). The remaining terms are nonnegative since row $k\oplus i$ of D is. This establishes $e_i^T d\mathcal{G}_{e_k}(p-e_k) > e_i^T(p-e_k)$.

Applying this inequality recursively yields $e_i^T d\mathcal{G}_{e_k}^j(p-e_k) > e_i^T(p-e_k)$ for all j. Thus $p - e_k$ can not lie in the stable space of $d\mathcal{G}_{e_k}$. ∎

We say a fixed point p of $\mathcal{G} : \Lambda \longrightarrow \Lambda$ is *unstable* if there exists a relative neighborhood[1] V in Λ of p such that for any neighborhood V' of p, there exists a $q \in V'$ and an integer t such that $\mathcal{G}^t(q) \notin V$.

[1] A relative neighborhood of p in Λ is the intersection of a neighborhood of p in R^n with Λ.

Theorem 4.3 *If the spectral radius of $d\mathcal{G}_{e_k}$ is greater than 1, then e_k is an unstable fixed point of $\mathcal{G}$, considered as a map from Λ to Λ.*

> **Proof:** We use the same notation and conventions as in the proof of Theorem 4.1. By Lemma 4.2, the translate by e_k of the stable space of $d\mathcal{G}_{e_k}$ does not intersect the interior of Λ. Choose norms for R^n as in the proof of Theorem 4.1 and let $p - e_k = s + u$ where p is in the interior of Λ. Then $\|u\| > 0$ and $p \in e_k + r(\theta)$ for some θ. Now for sufficiently small δ with $\delta(p - e_k) + e_k \in V$, there exists t such that $\mathcal{G}^t(\delta p) \notin V$ (here V is the neighborhood given by Theorem 4.1). ∎

The importance of theorem 4.3 is that if $\rho(d\mathcal{G}_{e_k}) > 1$, then populations arbitrarily close to e_k are expected to follow an evolutionary trajectory moving away from e_k. In fact, the proof shows all near by interior points of Λ are expected to behave in this way.

It appears as though the stability analysis just presented covers only the fixed points of $\mathcal{G}$ found at vertices of Λ. However, when mutation is zero, we do not know of any examples where stable fixed points are not at vertices.

Conjecture 4.4 *If mutation is zero, the only stable fixed points of $\mathcal{G}$ are at vertices of Λ.*

The *basin of attraction* of a fixed point x is the set of points whose trajectories converge to x. A fixed point is *hyperbolic* if the differential has no eigenvalues of modulus 1. For hyperbolic fixed points, the Stable Manifold Theorem shows that the translate of the stable space of the differential is tangent to the basin of attraction of x. Recall that Lemma 4.2 shows the translate of the stable space of the differential does not intersect the interior of Λ. This motivates the following generalization.

Conjecture 4.5 *If mutation is zero, the basin of attraction of an unstable vertex fixed point of $\mathcal{G}$ does not intersect the interior of the simplex.*

5 APPLICATIONS

In this section we specialize the formula of Theorem 3.4 to one-point and uniform crossover. Then an example is given of a “fully nondeceptive” fitness function that has exponentially many stable fixed points.

For one-point crossover, the crossover mask probabilities are

$$\chi_u = \begin{cases} 1-\chi & \text{if } u = 0 \\ \chi/(\ell-1) & \text{if } u = 2^k - 1 \text{ for some integer } k, 1 \le k < \ell \\ 0 & \text{otherwise} \end{cases}$$

where χ is the crossover rate. For uniform crossover, the crossover mask probabilities are given by $\chi_u = \delta_{u,0}(1-\chi)+\chi/n$. Here χ is used both as a vector, to specify the probability χ_i that crossover mask i si used, and as a scalar, to specify the crossover rate. This overloading of χ does not take long to get used to because context makes its meaning clear.

For $i \neq 0$, let $\mathrm{lo}(i)$ and $\mathrm{hi}(i)$ be the smallest and largest k such that $i \otimes 2^k \neq 0$. Note that $\mathrm{lo}(i)$ and $\mathrm{hi}(i)$ are the smallest and largest nonzero bit positions in i. Define $\delta(i) = \mathrm{hi}(i) - \mathrm{lo}(i) + 1$. The following lemma is a special case of a formula proved by Gary Koehler [3].

Lemma 5.1 *For $i \neq 0$ and one-point crossover,*

$$\sum_u (\chi_u + \chi_{\overline{u}})[u \otimes i = 0] = 1 - \chi + \chi \frac{\ell - \delta(i)}{\ell - 1}$$

Lemma 5.2 *For $i \neq 0$ and uniform crossover,*

$$\sum_u (\chi_u + \chi_{\overline{u}})[u \otimes i = 0] = 1 - \chi + \chi 2^{1-|i|}$$

where $|i| = \mathbf{1}^T i$ is the number of nonzero bits in i.

> **Proof:** The cardinality of the set $\{u : u \otimes i = 0\}$ is $2^{\ell-|i|}$. If $i \neq 0$, then the sets $\{u : u \otimes i = 0\}$ and $\{\overline{u} : u \otimes i = 0\}$ are disjoint. Hence their union has cardinality $2^{\ell-|i|+1}$. Thus
>
> $$\sum_u (\chi_u + \chi_{\overline{u}})[u \otimes i = 0] = 1 - \chi + 2^{\ell-|i|+1}\chi/n$$
>
> ∎

Two schema are said to be *competing* if they have the same fixed positions but different fixed bits. Let S_1 and S_2 be competing and suppose S_1 contains the maximum. A fitness function has been called *fully nondeceptive* if for every such pair, S_1 has the higher average fitness. We now give a family of fully nondeceptive fitness functions which have $2^{\ell-1}$ stable fixed points.

If $|k|$ is even (i.e., k has even parity), let $f_k = a$, and if $|k|$ is odd, let $f_k = b$ where $a > b > 0$. Any schema with more than one element contains equal numbers of even and odd parity strings. Hence, all nontrivial schemata have equal average fitness. Now modify f to $f_k = a + c(\ell - |k|)$ for even parity k. If $c > 0$ then 0 is the unique point of maximum fitness and f is fully nondeceptive. Under appropriate choice of a, b, and c, this function will have the required properties.

For any crossover type, define

$$h = \max_{|i|>1} \sum (\chi_u + \chi_{\overline{u}})[u \otimes i = 0]$$

In the case of one-point crossover, $h = 1 - \chi/(\ell - 1)$. For uniform crossover, $h = 1 - \chi/2$.

Proposition 5.3 *If $0 < c < \frac{a}{\ell}(\frac{1}{h} - 1)$, then f as defined above is fully nondeceptive. If $|k|$ is even, then e_k is a stable fixed point of $\mathcal{G}$ given any crossover for which $h < 1$.*

> **Proof:** The function f is fully nondeceptive by construction. If $|k|$ is even and $|i| > 1$, then
>
> $$\frac{f_{i \oplus k}}{f_k} \sum_u (\chi_u + \chi_{\overline{u}})[u \otimes i = 0] \quad \leq \quad \frac{a + c\,\ell}{a} h < 1$$
>
> If $|k|$ is even and $|i| = 1$, then $|k \oplus i|$ is odd, $f_{k \oplus i} = b < a = f_k$, and the summation above is 1. These considerations together with Theorem 3.4 show $\rho(d\mathcal{G}_{e_k}) < 1$. Hence e_k is stable.
>
> ∎

The next proposition shows that if fitnesses are distinct then $2^{\ell-1}$ is the maximum possible number of stable fixed points.

Proposition 5.4 *Suppose* $|j \oplus k| = 1$ *(*j *and* k *differ by exactly one bit). If* $f_j \neq f_k$*, then at most one of* e_j *and* e_k *is stable.*

> **Proof:** Without loss of generality, assume $f_j > f_k$. Let $i = j \oplus k$ and consider the eigenvalue of $d\mathcal{G}_{e_k}$ given by $\frac{f_{i \oplus k}}{f_k} \sum_u (\chi_u + \chi_{\overline{u}})[u \otimes i = 0]$. Since the summation is 1, the eigenvalue is $f_j / f_k > 1$, and e_k can not be stable. ∎

6 CONCLUSION

In the case of no mutation, we have defined $\mathcal{G}$, the infinite population Simple Genetic Algorithm. We have produced a formula for the spectrum of $d\mathcal{G}$ that allows the determination of the stability of fixed points at vertices of Λ. Fixed points are important because they represent populations towards which populations may evolve. When a fixed point p is stable, one would expect a GA could become trapped there. If p were unstable, it would be less likely that convergence to p would take place (for large populations).

Even though our method applies only to vertex fixed points (populations consisting of a single string type), we believe that all stable fixed points are in fact vertices when mutation is zero and fitnesses are distinct. We have made related conjectures concerning these matters (see conjectures 3.4 and 3.5) which identify important open questions.

In the final section, our analysis is applied to construct functions which are – from a "static schema analysis perspective" – totally easy, yet have exponentially many suboptimum stable fixed points. We also show these functions to be extremal in the sense of having the maximum number of stable fixed points possible.

Acknowledgements

This research was supported by the National Science Foundation IRI-9224917.

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